A.2

$S_0 \times S_1$ and $S_1 \times S_0$
Define $\rho$ to be a function from $S_0 \times S_1$ to $S_1 \times S_0$:
$$\rho = \{[(s_0, s_1), (s_1, s_0)] \mid s_0 \in S_0 \text{ and } s_1 \in S_1\}$$
Then
$$\rho^\dagger = \{[(s_1, s_0), (s_0, s_1)] \mid s_0 \in S_0 \text{ and } s_1 \in S_1\}$$
is a well defined function from $S_1 \times S_0$ to $S_0 \times S_1$, so $\rho$ is an isomorphism.

$(S_0 \times S_1) \times S_2$ and $S_0 \times (S_1 \times S_2)$
Define $\rho$ to be a function from $(S_0 \times S_1) \times S_2$ to $S_0 \times (S_1 \times S_2)$:
$$\rho = \{[(s_0, (s_1, s_2)), (s_0, (s_1, s_2))] \mid s_0 \in S_0, s_1 \in S_1 \text{ and } s_2 \in S_2\}$$
Then
$$\rho^\dagger = \{[(s_0, (s_1, s_2)), (s_0, (s_1, s_2))] \mid s_0 \in S_0, s_1 \in S_1 \text{ and } s_2 \in S_2\}$$
is a well defined function from $S_0 \times (S_1 \times S_2)$ to $(S_0 \times S_1) \times S_2$, so $\rho$ is an isomorphism.

$S_0 + S_1$ and $S_1 + S_0$
Define $\rho$ to be a function from $S_0 + S_1$ to $S_1 + S_0$:
$$\rho = \{[(0, x), (1, x)] \mid x \in S_0\} \cup \{[(1, x), (0, x)] \mid x \in S_1\}$$
Then
$$\rho^\dagger = \{[(1, x), (0, x)] \mid x \in S_0\} \cup \{[(0, x), (1, x)] \mid x \in S_1\}$$
is a well defined function from $S_1 + S_0$ to $S_0 + S_1$, so $\rho$ is an isomorphism.

$(S_0 + S_1) + S_2$ and $S_0 + (S_1 + S_2)$:
Define $\rho$ to be a function from $(S_0 + S_1) + S_2$ to $S_0 + (S_1 + S_2)$:
$$\rho = \{[(0, (0, x)), (0, x)] \mid x \in S_0\} \cup \{[(0, (1, x)), (1, (0, x))] \mid x \in S_1\} \cup \{[(1, x), (1, (1, x))] \mid x \in S_2\}$$
Then
$$\rho^\dagger = \{[(0, x), (0, (0, x))] \mid x \in S_0\} \cup \{[(1, (0, x)), (0, (1, x))] \mid x \in S_1\} \cup \{[(1, (1, x)), (1, x)] \mid x \in S_2\}$$
is a well defined function from $S_0 + (S_1 + S_2)$ to $(S_0 + S_1) + S_2$, so $\rho$ is an isomorphism.
A.3(b)

Define

\[ R = \{0 : 0, 0 : 1\} \]
\[ R' = \{0 : 1 | 1 : 1\} \]

Then

\[ (\cap R) \cdot (\cap R') = \{\} \cdot R' = \{\} \]

but

\[ \cap \{\rho \cdot \rho' | \rho \in R \text{ and } \rho' \in R\} = \cap \{[0 : 1], [0 : 1]\} = [0 : 1] \neq (\cap R) \cdot (\cap R') \]

A.5

Let

\[ \rho_1 = \{[n, 2n] | n \in \mathbb{N}\} \]
\[ \rho_2 = \{[n, 2n] | n \in \mathbb{N}\} \cup \{[n, 2n + 1] | n \in \mathbb{N}\} \]
\[ \rho_3 = \{[2n, 2n] | n \in \mathbb{N}\} \cup \{[2n, 2n + 1] | n \in \mathbb{N}\} \]
\[ \rho_4 = \{[2n, 2n + 1] | n \in \mathbb{N}\} \cup \{[2n + 1, 2n] | n \in \mathbb{N}\} \]

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\[ \rho_1 \cdot \rho_1 = \{[n, 4n] | n \in \mathbb{N}\} \]

2
\( \rho_2 \cdot \rho_2 = \{ [n, 4n] \mid n \in \mathbb{N} \} \cup \{ [n, 4n + 1] \mid n \in \mathbb{N} \} \cup \{ [n, 4n + 2] \mid n \in \mathbb{N} \} \cup \{ [n, 4n + 3] \mid n \in \mathbb{N} \} \)

\( \rho_3 \cdot \rho_3 = \rho_3 \)

\( \rho_4 \cdot \rho_4 = I_N \)

\begin{align*}
\rho_1^\dagger &= \{ [2n, n] \mid n \in \mathbb{N} \} \\
\rho_2^\dagger &= \{ [2n, n] \mid n \in \mathbb{N} \} \cup \{ [2n + 1, n] \mid n \in \mathbb{N} \} \\
\rho_3^\dagger &= \{ [2n, 2n] \mid n \in \mathbb{N} \} \cup \{ [2n + 1, 2n] \mid n \in \mathbb{N} \} \\
\rho_4^\dagger &= \rho_4 \\
\rho_1(x) &= 2x \\
\rho_4(x) &= \text{if even } x \text{ then } x + 1 \text{ else } x - 1 \\
\rho_2^\dagger(x) &= \text{if even } x \text{ then } x/2 \text{ else } (x - 1)/2 \\
\rho_3^\dagger(x) &= \text{if even } x \text{ then } x \text{ else } x - 1 \\
\rho_4^\dagger(x) &= \text{if even } x \text{ then } x + 1 \text{ else } x - 1
\end{align*}

**A.6(b)**

"\Rightarrow" By definition, for any pair \([x, y] \in \rho \cdot \rho^\dagger\), there should exists an \(x\) such that \([x, x'] \in \rho^\dagger\), which implies \([x', x] \in \rho\), and \([x', y] \in \rho\). Given that \(\rho\) is a partial function, \([x', x] \in \rho\) means \(y = x\). So all the pairs in \(\rho \cdot \rho^\dagger\) are of the form \([x, y] \in I_{S'}\).

"\Leftarrow" Suppose that \(\rho\) is not a partial function, i.e. exists \(x \in S\) and \(y_1, y_2 \in S', y_1 \neq y_2\), such that both \([x, y_1]\) and \([x, y_2]\) are in \(\rho\). Since \([y_1, x] \in \rho^\dagger\) and \([x, y_2] \in \rho\), \([y_1, y_2] \in \rho \cdot \rho^\dagger \not\subseteq I_{S'}\).

**A.7**

(a) Since \(z\) is an upper bound of \(\{x, y\}\) and \(z'\) is a least upper bound, by definition, \(z' \subseteq z\). Similarly, \(z'\) being an upper bound and \(z\) being a least upper bound, \(z \subseteq z'\). But since \(\subseteq\) is a partial order, it’s antisymmetric, and the only possibility for both of the relations hold is \(z = z'\).

(b) Since \(x\) is the least upper bound of \(X\), for all the upper bounds \(y \in Y\), \(x \subseteq y\), thus \(x\) is a lower bound of \(Y\). To prove that it’s the greatest one, consider a \(z\) which is a lower bound of \(Y\). Observe that \(x\) itself is also a upper bound of \(X\), so \(x \in Y\), which means that \(z \subseteq x\) holds since \(z\) is a lower bound of \(Y\). Because that \(z \subseteq x\) holds for arbitrary lower bound \(z\), \(x\) is the greatest lower bound of \(Y\).

(c) Firstly assume

\[
u = \bigcup \{ \sqcup X \mid X \in \mathcal{X} \}
\]

exists. Then for every \(x \in \sqcup \mathcal{X}\), there exists \(X \in \mathcal{X}\) such that \(x \in X\). But

\[
x \subseteq \sqcup X \subseteq u
\]
which means $x$ is also an upper bound of $\bigcup \mathcal{X}$. To prove that it’s the least one, suppose there is a $u'$ which is a upper bound of $\bigcup \mathcal{X}$. Then it is also upper bounds of all the $X \in \mathcal{X}$, so $\bigcup X \subseteq u'$. Again, it means that $u'$ is a upper bound of $\{ \bigcup X \mid X \in \mathcal{X} \}$, which means $u \subseteq u'$ as $u$ being the least upper bound of it.

On the other hand, assume

$$u' = \bigcup \mathcal{X}$$

exists. By the same reason in the previous paragraph, $u'$ is also an upper bound of $\{ \bigcup X \mid X \in \mathcal{X} \}$. Suppose there is a $u$ which is an upper bound of $\{ \bigcup X \mid X \in \mathcal{X} \}$, we can show, as in the previous paragraph, that $u$ is also an upper bound of $\bigcup \mathcal{X}$ and by the leastness concludes that $u' \subseteq u$ and $u'$ is the least upper bound of $\{ \bigcup X \mid X \in \mathcal{X} \}$.

1.2

(a) $\exists c. a \times c = b$

(b) $\exists b'. \exists c'. a \times b' = b \land a \times c' = c$

(c) $\exists b'. \exists c'. a \times b' = b \land a \times c' = c \land \forall a'. (\exists b''. \exists c''. a' \times b'' = b \land a' \times b'' = b) \Rightarrow a' \leq a$

(d) $\forall a. \forall b. a \times b = p \Rightarrow (a = 1 \lor b = 1)$ or

$\neg p = 1 \land \forall a. (\exists b. a \times b = p) \Rightarrow (a = 1 \lor a = p)$