CPSC 629: Reading Group
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Homework 1

Due: Monday, February 23, 2004.

Consider the formulation of the Hilbert and the natural deduction calculus from class. The homework problem consists of extending the two system by rules for the existential and universal quantifier. Furthermore, you are asked to extend the three theorems that we discussed in class, i.e. the proof of the deduction theorem, the embedding of derivations in the natural deduction system into the Hilbert system, and the reverse embeding of Hilbert derivations back into the natural deduction calculus. Here are the rules that extend the natural deduction calculus.

$$\frac{\overset{!}{\vdash} F[a/x]}{\overset{!}{\vdash} \forall x.F}\,\forall I^a \qquad \frac{\overset{!}{\vdash} \forall x.F}{\overset{!}{\vdash} F[t/x]}\,\forall E \qquad \frac{\overset{!}{\vdash} F[t/x]}{\overset{!}{\vdash} \exists x.F}\,\exists I \qquad \frac{\overset{!}{\vdash} \exists x.F \qquad \begin{array}{c} \overline{\overset{!}{\vdash} F[a/x]}\;u \\ \vdots \\ \overset{!}{\vdash} C \end{array}}{\overset{!}{\vdash} C}\,\exists E^{a,u}$$

Two more axioms and two more inference rules extend our formulation of the Hilbert calculus.

$$\frac{}{\vdash \forall x.F \supset F[t/x]}\,\forall_1 \qquad \frac{\vdash G \supset F[a/x]}{\vdash G \supset \forall x.F}\,\forall_2^a \qquad \frac{}{\vdash F[t/x] \supset \exists x.F}\,\exists_1 \qquad \frac{\vdash F[a/x] \supset G}{\vdash \exists x.F \supset G}\,\exists_2^a$$

Exercise 1 (Representation)

1. Give an encoding of the previous eight rules in LF.
2. Show the adequacy of your encoding.

Exercise 2 (Reasoning)

1. Prove the resulting new cases for the deduction theorem

   Theorem 1 A proof of $\vdash G$ from assumption $\vdash F$ can be converted into a proof of $\vdash F \supset G$.

2. Prove the resulting new cases for the embedding theorem

   Theorem 2 Any proof of $\overset{!}{\vdash} F$ can be converted into a proof of $\vdash F$.

3. Prove the resulting new cases for the reverse direction of the embedding theorem

Theorem 3 Any proof of $\vdash F$ can be converted into a proof of $\overset{!}{\vdash} F$.

Exercise 3 (Implementation) Implement your proof in LF. Extend the following three type families by the new cases.

1. `ded: (hil F -> hil G) -> hil (F => G) -> type.`
2. `ndhil: nd F -> hil F -> type.`
3. `hilnd: hil F -> nd F -> type.`

Argue in each case why your implementation represents a proof.