CS 430: Formal Semantics Assignment 1 Sample Solution

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A.2

$S_0 \times S_1$ and $S_1 \times S_0$

Define ρ to be a function from $S_0 \times S_1$ to $S_1 \times \S_0$:

$$\rho = \{[\langle s_0, s_1\rangle, \langle s_1, s_0\rangle] \mid s_0 \in S_0 \text{ and } s_1 \in S_1\}$$

Then

$$\rho^\dagger = \{[\langle s_1, s_0\rangle, \langle s_0, s_1\rangle] \mid s_0 \in S_0 \text{ and } s_1 \in S_1\}$$

is a well defined function from $S_1 \times S_0$ to $S_0 \times S_1$, so ρ is an isomorphism.

$(S_0 \times S_1) \times S_2$ and $S_0 \times (S_1 \times S_2)$

Define ρ to be a function from $(S_0 \times S_1) \times S_2$ to $S_0 \times (S_1 \times S_2)$:

$$\rho = \{[\langle\langle s_0, s_1\rangle, s_2\rangle, \langle s_0, \langle s_1, s_2\rangle\rangle] \mid s_0 \in S_0, s_1 \in S_1 \text{ and } s_2 \in S_2\}$$

Then

$$\rho^\dagger = \{[\langle s_0, \langle s_1, s_2\rangle\rangle, \langle\langle s_0, s_1\rangle, s_2\rangle] \mid s_0 \in S_0, s_1 \in S_1 \text{ and } s_2 \in S_2\}$$

is a well defined function from $S_0 \times (S_1 \times S_2)$ to $(S_0 \times S_1) \times S_2$, so ρ is an isomorphism.

$S_0 + S_1$ and $S_1 + S_0$

Define ρ to be a function from $S_0 + S_1$ to $S_1 + S_0$:

$$\rho = \{[\langle 0, x\rangle, \langle 1, x\rangle] \mid x \in S_0\} \cup \{[\langle 1, x\rangle, \langle 0, x\rangle] \mid x \in S_1\}$$

Then

$$\rho^\dagger = \{[\langle 1, x\rangle, \langle 0, x\rangle] \mid x \in S_0\} \cup \{[\langle 0, x\rangle, \langle 1, x\rangle] \mid x \in S_1\}$$

is a well defined function from $S_1 + S_0$ to $S_0 + S_1$, so ρ is an isomorphism.

$(S_0 + S_1) + S_2$ and $S_0 + (S_1 + S_2)$:

Define ρ to be a function from $(S_0 + S_1) + S_2$ to $S_0 + (S_1 + S_2)$:

$$\rho = \{[\langle 0, \langle 0, x\rangle\rangle, \langle 0, x\rangle] \mid x \in S_0\} \cup \{[\langle 0, \langle 1, x\rangle\rangle, \langle 1, \langle 0, x\rangle\rangle] \mid x \in S_1\} \cup \{[\langle 1, x\rangle, \langle 1, \langle 1, x\rangle\rangle] \mid x \in S_2\}$$

Then

$$\rho^\dagger = \{[\langle 0, x\rangle, \langle 0, \langle 0, x\rangle\rangle] \mid x \in S_0\} \cup \{[\langle 1, \langle 0, x\rangle\rangle, \langle 0, \langle 1, x\rangle\rangle] \mid x \in S_1\} \cup \{[\langle 1, \langle 1, x\rangle\rangle, \langle 1, x\rangle] \mid x \in S_2\}$$

is a well defined function from $S_0 + (S_1 + S_2)$ to $(S_0 + S_1) + S_2$, so ρ is an isomorphism.

A.3(b)

Define

$$R = \{[0:0], [0:1]\}$$
$$R' = \{[0:1 \mid 1:1]\}$$

Then

$$(\cap R) \cdot (\cap R') = \{\} \cdot R' = \{\}$$

but

$$\cap\{\rho \cdot \rho' \mid \rho \in R \text{ and } \rho' \in R\} = \cap\{[0:1], [0:1]\} = [0:1] \neq (\cap R) \cdot (\cap R')$$

A.5

Let

$$\begin{aligned}
\rho_1 &= \{[n, 2n] \mid n \in \mathbf{N}\} \\
\rho_2 &= \{[n, 2n] \mid n \in \mathbf{N}\} \cup \{[n, 2n+1] \mid n \in \mathbf{N}\} \\
\rho_3 &= \{[2n, 2n] \mid n \in \mathbf{N}\} \cup \{[2n, 2n+1] \mid n \in \mathbf{N}\} \\
\rho_4 &= \{[2n, 2n+1] \mid n \in \mathbf{N}\} \cup \{[2n+1, 2n] \mid n \in \mathbf{N}\}
\end{aligned}$$

	ρ_1	ρ_2	ρ_3	ρ_4
Total	Y	Y	N	Y
Partial function	Y	N	N	Y
Function	Y	N	N	Y
Surjection	N			Y
Injection	Y			Y
Bijection	N			Y
Transitive	N	N	Y	N
Symmetric	N	N	N	Y
Antisymmetric	Y	Y	Y	N
Reflexive	N	N	N	N
Preorder	N	N	N	N
Partial order	N	N	N	N
Equivalence	N	N	N	N
Partial equivalence	N	N	N	N
$\rho^\dagger$ total	N	Y	Y	Y
$\rho^\dagger$ partial function	Y	Y	Y	Y
$\rho^\dagger$ function	N	Y	Y	Y
$\rho^\dagger$ surjection		Y	N	Y
$\rho^\dagger$ injection		N	N	Y
$\rho^\dagger$ bijection		N	N	Y

$$\rho_1 \cdot \rho_1 = \{[n, 4n] \mid n \in \mathbf{N}\}$$

$$
\begin{array}{rcl}
\rho_2 \cdot \rho_2 & = & \{[n, 4n] \mid n \in \mathbf{N}\} \cup \{[n, 4n+1] \mid n \in \mathbf{N}\} \cup \{[n, 4n+2] \mid n \in \mathbf{N}\} \cup \{[n, 4n+3] \mid n \in \mathbf{N}\} \\
\rho_3 \cdot \rho_3 & = & \rho_3 \\
\rho_4 \cdot \rho_4 & = & I_{\mathbf{N}} \\
\hline
\rho_1^\dagger & = & \{[2n, n] \mid n \in \mathbf{N}\} \\
\rho_2^\dagger & = & \{[2n, n] \mid n \in \mathbf{N}\} \cup \{[2n+1, n] \mid n \in \mathbf{N}\} \\
\rho_3^\dagger & = & \{[2n, 2n] \mid n \in \mathbf{N}\} \cup \{[2n+1, 2n] \mid n \in \mathbf{N}\} \\
\rho_4^\dagger & = & \rho_4 \\
\hline
\rho_1(x) & = & 2x \\
\rho_4(x) & = & \textbf{if even } x \textbf{ then } x+1 \textbf{ else } x-1 \\
\rho_2^\dagger(x) & = & \textbf{if even } x \textbf{ then } x/2 \textbf{ else } (x-1)/2 \\
\rho_3^\dagger(x) & = & \textbf{if even } x \textbf{ then } x \textbf{ else } x-1 \\
\rho_4^\dagger(x) & = & \textbf{if even } x \textbf{ then } x+1 \textbf{ else } x-1
\end{array}
$$

A.6(b)

"$\Rightarrow$" By definition, for any pair $[x, y] \in \rho \cdot \rho^\dagger$, there should exists an x such that $[x, x'] \in \rho^\dagger$, which implies $[x', x] \in \rho$, and $[x', y] \in \rho$. Given that ρ is a partial function, $[x', y] \in \rho$ and $[x', x] \in \rho$ means $y = x$. So all the pairs in $\rho \cdot \rho^\dagger$ are of the form $[x, x]$, $\rho \cdot \rho^\dagger \subseteq I_{S'}$.

"$\Leftarrow$" Suppose that ρ is not a partial function, i.e. exists $x \in S$ and $y_1, y_2 \in S'$, $y_1 \neq y_2$, such that both $[x, y_1]$ and $[x, y_2]$ are in ρ. Since $[y_1, x] \in \rho^\dagger$ and $[x, y_2] \in \rho$, $[y_1, y_2] \in \rho \cdot \rho^\dagger \nsubseteq I_{S'}$.

A.7

(a) Since z is an upper bound of $\{x, y\}$ and z' is a least upper bound, by definition, $z' \sqsubseteq z$. Similarly, z' being an upper bound and z being a least upper bound, $z \sqsubseteq z'$. But since $\sqsubseteq$ is a partial order, it's antisymmetric, and the only possibility for both of the relations hold is $z = z'$.

(b) Since x is the least upper bound of X, for all the upper bounds $y \in Y$, $x \sqsubseteq y$, thus x is a lower bound of Y. To prove that it's the greatest one, consider a z which is a lower bound of Y. Observe that x itself is also a upper bound of X, so $x \in Y$, which means that $z \sqsubseteq x$ holds since z is a lower bound of Y. Because that $z \sqsubseteq x$ holds for arbitrary lower bound z, x is the greatest lower bound of Y.

(c) Firstly assume

$$u = \bigsqcup \{\sqcup X \mid X \in \mathcal{X}\}$$

exists. Then for every $x \in \cup \mathcal{X}$, there exists $X \in \mathcal{X}$ such that $x \in X$. But

$$x \sqsubseteq \sqcup X \sqsubseteq u$$

which means x is also an upper bound of $\cup\mathcal{X}$. To prove that it's the least one, suppose there is a u' which is a upper bound of $\cup\mathcal{X}$. Then it is also upper bounds of all the $X \in \mathcal{X}$, so $\sqcup X \sqsubseteq u'$. Again, it means that u' is a upper bound of $\{\sqcup X \mid X \in \mathcal{X}\}$, which means $u \sqsubseteq u'$ as u being the least upper bound of it.

On the other hand, assume

$$u' = \bigsqcup \cup \mathcal{X}$$

exists. By the same reason in the previous paragraph, u' is also an upper bound of $\{\sqcup X \mid X \in \mathcal{X}\}$. Suppose there is a u which is an upper bound of $\{\sqcup X \mid X \in \mathcal{X}\}$, we can show, as in the previous paragraph, that u is also an upper bound of $\sqcup\cup\mathcal{X}$ and by the leastness concludes that $u' \sqsubseteq u$ and u' is the least upper bound of $\{\sqcup X \mid X \in \mathcal{X}\}$.

1.2

(a) $\exists c.\mathtt{a} \times c = \mathtt{b}$

(b) $\exists b'.\exists c'.\mathtt{a} \times b' = \mathtt{b} \wedge \mathtt{a} \times c' = \mathtt{c}$

(c) $\exists b'.\exists c'.\mathtt{a} \times b' = \mathtt{b} \wedge \mathtt{a} \times c' = \mathtt{c} \wedge \forall a'.(\exists b''.\exists c''.a' \times b'' = \mathtt{b} \wedge a' \times b'' = \mathtt{b}) \Rightarrow a' \leq \mathtt{a}$

(d) $\forall a.\forall b.a \times b = \mathtt{p} \Rightarrow (a = 1 \vee b = 1)$ or
$\neg\mathtt{p} = 1 \wedge \forall a.(\exists b.a \times b = \mathtt{p}) \Rightarrow (a = 1 \vee a = p)$