CS 430: Formal Semantics Assignment 2 Sample Solution

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Reynolds, exercise 2.2

(a) Syntax of commands is extended as follows:

$$c ::= \cdots \mid \textbf{repeat}\ c\ \textbf{until}\ b$$

The semantic function for this command is:

$$\begin{aligned} [\![\textbf{repeat}\ c\ \textbf{until}\ b]\!]_{\text{comm}} &= (\mathbf{Y}_{\Sigma\to\Sigma_\perp} F)_{\perp\!\!\!\perp} \cdot [\![c]\!]_{\text{comm}} \\ \text{where } F f\ \sigma &= \texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ \sigma\ \texttt{else}\ f_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \end{aligned}$$

(b)

$$[\![\textbf{repeat}\ c\ \textbf{until}\ b]\!]_{\text{comm}} = [\![c; \textbf{while}\ \neg b\ \textbf{do}\ c]\!]_{\text{comm}}$$

(c) From (a), we have:

$$\begin{aligned} F &= \lambda f.\lambda\sigma.\texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ \sigma\ \texttt{else}\ f_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \\ &= \lambda f.\lambda\sigma.\texttt{if}\ [\![\neg b]\!]_{\text{bool}}\sigma\ \texttt{then}\ f_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma)\ \texttt{else}\ \sigma \\ &=_{\text{def}} G \end{aligned}$$

Thus $(\mathbf{Y}_{\Sigma\to\Sigma_\perp} F) = (\mathbf{Y}_{\Sigma\to\Sigma_\perp} G)$ by uniqueness of least fixed-point. But note that $(\mathbf{Y}_{\Sigma\to\Sigma_\perp} G)$ is exactly the same to $[\![\textbf{while}\ \neg b\ \textbf{do}\ c]\!]_{\text{comm}}$ by definition of the semantics for **while**. Therefore,

$$\begin{aligned} [\![\textbf{repeat}\ c\ \textbf{until}\ b]\!]_{\text{comm}}\sigma &= (\mathbf{Y}_{\Sigma\to\Sigma_\perp} F)_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \\ &= (\mathbf{Y}_{\Sigma\to\Sigma_\perp} G)_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \\ &= [\![\textbf{while}\ \neg b\ \textbf{do}\ c]\!]_{\text{comm}\,\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \\ &= [\![c; \textbf{while}\ \neg b\ \textbf{do}\ c]\!]_{\text{comm}}\sigma \end{aligned}$$

Reynolds, exercise 2.3

The F used to define the semantics of the while loop is defined as

$$\begin{aligned} F\ f\ \sigma &= \texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ f_{\perp\!\!\!\perp}([\![c]\!]_{\text{comm}}\sigma)\ \texttt{else}\ \sigma \\ &= \texttt{if}\ \sigma x \neq 0\ \texttt{then}\ f[\sigma \mid x : \sigma x - 2]\ \texttt{else}\ \sigma \end{aligned}$$

We shall prove by induction that, for $n \geq 0$,

$$F^n \perp \sigma = \texttt{if}\ \texttt{even}(\sigma x) \wedge 0 \leq \sigma x < 2n\ \texttt{then}\ [\sigma \mid x : 0]\ \texttt{else}\ \perp$$

Base case: for $n = 0$, LHS $= I \perp \sigma = \perp$ and the condition $0 \leq \sigma x < 0$ is never true, hence RHS $= \perp =$ LHS.

Induction step: suppose that F^n is as claimed, consider F^{n+1}:

$$\begin{array}{rcl} F^{n+1}\bot\ \sigma & = & F\ (F^n\bot)\ \sigma \\ & = & \texttt{if}\ \sigma x \neq 0\ \texttt{then}\ F^n\bot[\sigma \mid x : \sigma x - 2]\ \texttt{else}\ \sigma \\ & = & \texttt{if}\ \sigma x \neq 0\ \texttt{then} \\ & & \quad \texttt{if even}([\sigma \mid x : \sigma x - 2]x) \wedge 0 \leq [\sigma \mid x : \sigma x - 2]x < 2n\ \texttt{then}\ [\sigma \mid x : \sigma x - 2 \mid x : 0]\ \texttt{else}\ \bot \\ & & \texttt{else}\ \sigma \\ & = & \texttt{if}\ \sigma x \neq 0\ \texttt{then} \\ & & \quad \texttt{if even}(\sigma x - 2) \wedge 0 \leq \sigma x - 2 < 2n\ \texttt{then}\ [\sigma \mid x : 0]\ \texttt{else}\ \bot \\ & & \texttt{else}\ [\sigma \mid x : 0] \\ & = & \texttt{if}\ (\texttt{even}(\sigma x) \wedge 2 \leq \sigma x < 2n+2) \vee x = 0\ \texttt{then}\ [\sigma \mid x : 0]\ \texttt{else}\ \bot \\ & = & \texttt{if}\ (\texttt{even}(\sigma x) \wedge 0 \leq \sigma x < 2n+2)\ \texttt{then}\ [\sigma \mid x : 0]\ \texttt{else}\ \bot \end{array}$$

Then, with the property of the least fix-point,

$$\begin{array}{rcl} \mathbf{Y}_{\Sigma\to\Sigma_\bot} F\ \sigma & = & (\bigsqcup_{n=0}^{\infty} F^n\bot)\ \sigma \\ & = & \bigsqcup_{n=0}^{\infty} (F^n\bot\ \sigma) \qquad \text{By Prop. 2.2} \\ & = & \texttt{if even}(\sigma x) \wedge \sigma x \geq 0\ \texttt{then}\ [\sigma \mid x : 0]\ \texttt{else}\ \bot \end{array}$$

Reynolds, exercise 2.4

We need to show that $F = \lambda f.\lambda\sigma.\ \texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ f_{\bot\!\!\bot}([\![c]\!]_{\text{comm}}\sigma)\ \texttt{else}\ \sigma$ is continues. We begin by showing that F is monotone, that is, for all $f \sqsubseteq f'$, $Ff \sqsubseteq Ff'$, or $\forall\sigma, Ff\ \sigma \sqsubseteq Ff'\ \sigma$. This is simple to show by splitting cases on $[\![b]\!]_{\text{bool}}\sigma$, and noting that $f_{\bot\!\!\bot} \sqsubseteq f'_{\bot\!\!\bot}$ in the case when $[\![b]\!]_{\text{bool}}\sigma = \texttt{true}$.

Now to show that F is continuous, we need to show that for a chain $f_0 \sqsubseteq f_1 \sqsubseteq \cdots$,

$$\begin{array}{ll} & F(\bigsqcup_{i=0}^{\infty} f_i) = \bigsqcup_{i=0}^{\infty}(Ff_i) \\ \Leftrightarrow & \forall\sigma.F(\bigsqcup_{i=0}^{\infty} f_i)\sigma = (\bigsqcup_{i=0}^{\infty}(Ff_i))\sigma \\ \Leftrightarrow & \forall\sigma.F(\bigsqcup_{i=0}^{\infty} f_i)\sigma = \bigsqcup_{i=0}^{\infty}(Ff_i\ \sigma) \qquad \text{because of monotonicity and Prop. 2.2} \\ \Leftrightarrow & \forall\sigma.\texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ (\bigsqcup_{i=0}^{\infty} f_i)_{\bot\!\!\bot}([\![c]\!]_{\text{comm}}\sigma)\ \texttt{else}\ \sigma = \bigsqcup_{i=0}^{\infty}\texttt{if}\ [\![b]\!]_{\text{bool}}\sigma\ \texttt{then}\ (f_i)_{\bot\!\!\bot}([\![c]\!]_{\text{comm}}\sigma)\ \texttt{else}\ \sigma \end{array}$$

This again is simple to prove by splitting cases on $[\![b]\!]_{\text{bool}}\sigma$. If it is equal to false, then the LHS evaluates to

σ, while the RHS evaluates to $\bigsqcup \sigma = \sigma$. If it is equal to true, then:

$$\begin{aligned}
\text{LHS} &= (\bigsqcup_{i=0}^{\infty} f_i)_{\perp\!\!\perp}([\![c]\!]_{\text{comm}}\sigma) \\
&= (\bigsqcup_{i=0}^{\infty} (f_i)_{\perp\!\!\perp})([\![c]\!]_{\text{comm}}\sigma) && \text{by continuity of } (-)_{\perp\!\!\perp} \\
&= (\bigsqcup_{i=0}^{\infty} (f_i)_{\perp\!\!\perp}([\![c]\!]_{\text{comm}}\sigma)) && \text{by Prop. 2.2} \\
&= \text{RHS}
\end{aligned}$$

Reynolds, exercise 2.9

$$\begin{aligned}
&\textbf{for } v := e_0 \textbf{ to } e_1 \textbf{ do } c \stackrel{\text{def}}{=} \\
&\quad \textbf{newvar } v := e_0 \textbf{ in newvar } w := e_1 \textbf{ in while } v < w \textbf{ do } (c; v := v + 1) \textbf{ if } v = w \textbf{ then } c \textbf{ else skip}
\end{aligned}$$