Constructive Logic and Classical Logic

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Is $P = NP$?

- True
- False
- Currently unknown

Is $P = NP$?

- Prove $P = NP$
- Prove $P \neq NP$
- Neither $P = NP$ or $P \neq NP$ are provable

Gödel's incompleteness theorem

For any interesting axiomatic system, there are sentences of the system ϕ *for which there is no proof of* ϕ *or of* $\neg\phi$ *within the system.*

Constructive logic

- Reject the fact that every sentence is either *true* or *false*
- Perceive truth in terms of existence of proof:
  ϕ *is true* $\equiv$ *there is a proof of* ϕ
  ϕ *is false* $\equiv$ *a proof of* ϕ *leads to contradiction*
- Corresponds to mathematical practice
- Philosophically: no extrinsic notion of truth

Constructive logic

Based on these ideas, the rules of logic codify what counts as valid justification for a sentence.

Rules of Constructive Logic

Judgements

ϕ prop	(valid proposition)
$\Gamma \vdash \phi$ true	(ϕ has a proof)

Propositions

$$\phi ::= \top \mid \bot \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \supset \phi_2$$

$$\Gamma ::= \phi_1 \text{ true}, \cdots , \phi_n \text{ true}$$

Rules

Intro: direct evidence for a connective
Elim: use the existence of the proof to prove something else indirectly

Rules

Structural $\dfrac{\phi \text{ true} \in \Gamma}{\Gamma \vdash \phi \text{ true}}$

Truth $\dfrac{}{\Gamma \vdash \top \text{ true}}$ no elim rule

False no intro rule $\dfrac{\Gamma \vdash \bot \text{ true}}{\Gamma \vdash \phi \text{ true}}$

Rules

Conjuction

$$\frac{\Gamma \vdash \phi_1 \text{ true} \qquad \Gamma \vdash \phi_2 \text{ true}}{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ true}}$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ true}}{\Gamma \vdash \phi_1 \text{ true}} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ true}}{\Gamma \vdash \phi_2 \text{ true}}$$

Disjunction

$$\frac{\Gamma \vdash \phi_1 \text{ true}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ true}} \qquad \frac{\Gamma \vdash \phi_2 \text{ true}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ true}}$$

$$\frac{\begin{array}{c}\Gamma \vdash \phi_1 \vee \phi_2 \text{ true} \\ \Gamma,\ \phi_1 \text{ true} \vdash \phi \text{ true} \qquad \Gamma,\ \phi_2 \text{ true} \vdash \phi \text{ true}\end{array}}{\Gamma \vdash \phi \text{ true}}$$

Rules

Implication

$$\frac{\Gamma,\ \phi_1\ \mathsf{true} \vdash \phi_2\ \mathsf{true}}{\Gamma \vdash \phi_1 \supset \phi_2\ \mathsf{true}}$$

$$\frac{\Gamma \vdash \phi_1 \supset \phi_2\ \mathsf{true} \qquad \Gamma \vdash \phi_1\ \mathsf{true}}{\Gamma \vdash \phi_2\ \mathsf{true}}$$

$$\neg\phi \equiv \phi \supset \bot$$

Propositions as types

- The outermost connective of ϕ specifies the form of a valid proof
- e.g. a proof of $\phi_1 \vee \phi_2$: choose left or right, and provide a witness
- correspondence with terms of a programming language
- proofs as terms, propositions as types
- proving and programming is the same!

Term assignment

$$\phi_1 \text{ true}, \cdots, \phi_n \text{ true} \vdash \phi \text{ true}$$

becomes

$$x_1 : \phi_1, \cdots, x_n : \phi_n \vdash p : \phi$$

Rules

Structural

$$\frac{x : \phi \in \Gamma}{\Gamma \vdash x : \phi}$$

Truth

$$\frac{}{\Gamma \vdash \langle\rangle : \top}$$

no elim rule

False

no intro rule

$$\frac{\Gamma \vdash p : \bot}{\Gamma \vdash \text{abort } p : \phi}$$

Rules

Conjuction

$$\frac{\Gamma \vdash p_1 : \phi_1 \qquad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash \langle p_1, p_2 \rangle : \phi_1 \wedge \phi_2} \qquad \frac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \mathsf{fst}\ p : \phi_1}$$

$$\frac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \mathsf{snd}\ p : \phi_2}$$

Disjunction

$$\frac{\Gamma \vdash p_1 : \phi_1}{\Gamma \vdash \mathsf{inl}\ p : \phi_1 \vee \phi_2} \qquad \frac{\Gamma \vdash p_2 : \phi_2}{\Gamma \vdash \mathsf{inr}\ p : \phi_1 \vee \phi_2}$$

$$\frac{\Gamma \vdash p : \phi_1 \vee \phi_2 \qquad \Gamma,\ x : \phi_1 \vdash p_1 : \phi \qquad \Gamma,\ x : \phi_2 \vdash p_2 : \phi}{\Gamma \vdash \mathsf{case}(p; x.p_1; x.p_2) : \phi}$$

Rules

Implication

$$\frac{\Gamma,\ x:\phi_1 \vdash p:\phi_2}{\Gamma \vdash \lambda x:\phi_1.p:\phi_1 \supset \phi_2}$$

$$\frac{\Gamma \vdash p_1:\phi_1 \supset \phi_2 \qquad \Gamma \vdash p_2:\phi_1}{\Gamma \vdash p_1\ p_2:\phi_2}$$

$$\neg\phi \equiv \phi \supset \bot$$

Propositions-as-types correspondence
AKA Curry-Howard Isomorphism

Proposition	Type
$\top$	unit
$\bot$	void
$\phi_1 \wedge \phi_2$	$\tau_1 \times \tau_2$
$\phi_1 \vee \phi_2$	$\tau_1 + \tau_2$
$\phi_1 \supset \phi_2$	$\tau_1 \to \tau_2$
$\forall x.\phi$	?
$\exists x.\phi$	?

What about reduction?

When we view p as programs, we can evaluate them based on their operational semantics. What does this evaluation correspond to?

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When we view p as programs, we can evaluate them based on their operational semantics. What does this evaluation correspond to?

$$\dfrac{\dfrac{\Gamma,\ x : \phi_1 \vdash t : \phi_2}{\Gamma \vdash \lambda x : \phi_1 . t : \phi_1 \supset \phi_2} \qquad \dfrac{\cdots}{\Gamma \vdash t' : \phi_1}}{\Gamma \vdash (\lambda x : \phi_1 . t)\ t' : \phi_2} \leadsto \dfrac{\cdots}{\Gamma \vdash t[t'/x] : \phi_2}$$

Soundness

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- Is there a proof of $\bot$?
- If only neutral/canonical terms, easy
- Show that terms can always be reduced to canonical terms (normalization)

Soundness

- soundness is thus justified by strong/weak normalization
- reduction procedures (like hereditary substitutions) correspond to cut elimination procedures

Classical Logic

A classical proof

Theorem

$\exists a, b \in R.\mathit{irrational}(a) \wedge \mathit{irrational}(b) \wedge \mathit{rational}(a^b)$

Proof.

Consider $\sqrt{2}^{\sqrt{2}}$. This number is either rational or irrational. Suppose it is rational: then $a = \sqrt{2}, b = \sqrt{2}$ gives the required result. Suppose it is not: then $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ gives the required result, as $a^b = 2$. □

Example from Lecture Notes on Classical Logic, William Lovas and Frank Pfenning

Classical Logic

- weaken notion of truth of ϕ: instead of existence of proof for ϕ, existence of refutation for $\neg\phi$
- a classically valid proposition ϕ is irrefutable constructively
- symmetry between truth and falsity (false is existence of proof for $\neg\phi$)

Definition of Classical Logic

- could just add a classical axiom (e.g. excluded middle or double negation) to constructive logic
- instead: proper judgemental system to bring out the symmetry

Judgements of Classical Logic

$\Delta;\ \Gamma \vdash \phi$ true	the proposition ϕ is provable
$\Delta;\ \Gamma \vdash \phi$ false	the proposition ϕ is refutable
$\Delta;\ \Gamma \vdash \#$	a contradiction has been derived

$$\Delta ::= \phi_1 \text{ false}, \cdots, \phi_m \text{ false}$$
$$\Gamma ::= \phi_1 \text{ true}, \cdots, \phi_n \text{ true}$$

Rules

Truth rules: direct proof
Falsity rules: direct refutation
Contradiction rules: indirect proofs and refutations

Rules

Structural

$$\frac{\phi \text{ true} \in \Gamma}{\Delta; \Gamma \vdash \phi \text{ true}} \qquad \frac{\phi \text{ false} \in \Delta}{\Delta; \Gamma \vdash \phi \text{ false}}$$

Truth

$$\frac{}{\Delta; \Gamma \vdash \top \text{ true}}$$

no refutation

False

no proof

$$\frac{}{\Delta; \Gamma \vdash \bot \text{ false}}$$

Rules

Conjunction

$$\frac{\Delta; \Gamma \vdash \phi_1 \text{ true} \qquad \Delta; \Gamma \vdash \phi_2 \text{ true}}{\Delta;\ \Gamma \vdash \phi_1 \wedge \phi_2 \text{ true}}$$

$$\frac{\Delta; \Gamma \vdash \phi_1 \text{ false}}{\Delta; \Gamma \vdash \phi_1 \wedge \phi_2 \text{ false}} \qquad \frac{\Delta; \Gamma \vdash \phi_2 \text{ false}}{\Delta; \Gamma \vdash \phi_1 \wedge \phi_2 \text{ false}}$$

Disjunction

$$\frac{\Delta; \Gamma \vdash \phi_1 \text{ true}}{\Delta; \Gamma \vdash \phi_1 \vee \phi_2 \text{ true}} \qquad \frac{\Delta; \Gamma \vdash \phi_2 \text{ true}}{\Delta; \Gamma \vdash \phi_1 \vee \phi_2 \text{ true}}$$

$$\frac{\Delta; \Gamma \vdash \phi_1 \text{ false} \qquad \Delta; \Gamma \vdash \phi_2 \text{ false}}{\Delta;\ \Gamma \vdash \phi_1 \vee \phi_2 \text{ false}}$$

Rules

Implication

$$\frac{\Delta;\Gamma,\ \phi_1 \ \mathsf{true} \vdash \phi_2 \ \mathsf{true}}{\Delta;\Gamma \vdash \phi_1 \supset \phi_2 \ \mathsf{true}}$$

$$\frac{\Delta;\Gamma \vdash \phi_1 \ \mathsf{true} \qquad \Delta;\Gamma \vdash \phi_2 \ \mathsf{false}}{\Delta;\Gamma \vdash \phi_1 \supset \phi_2 \ \mathsf{false}}$$

Negation

$$\frac{\Delta;\Gamma \vdash \phi \ \mathsf{false}}{\Delta;\Gamma \vdash \neg\phi \ \mathsf{true}} \qquad \frac{\Delta;\Gamma \vdash \phi \ \mathsf{true}}{\Delta;\Gamma \vdash \neg\phi \ \mathsf{false}}$$

Rules

$$\frac{\Delta; \Gamma \vdash \phi \ \mathsf{true} \qquad \Delta; \Gamma \vdash \phi \ \mathsf{false}}{\Delta; \Gamma \vdash \#}$$

$$\frac{\Delta, u \ \mathsf{false}; \Gamma \vdash \#}{\Delta; \Gamma \vdash \phi \ \mathsf{true}}$$

$$\frac{\Delta; \Gamma, x \ \mathsf{true} \vdash \#}{\Delta; \Gamma \vdash \phi \ \mathsf{false}}$$

Term assignment

$\Delta;\ \Gamma \vdash \phi$ true	becomes	$\Delta;\ \Gamma \vdash p : \phi$
$\Delta;\ \Gamma \vdash \phi$ false	becomes	$\Delta;\ \Gamma \vdash k \div \phi$
$\Delta;\ \Gamma \vdash \#$	becomes	$\Delta;\ \Gamma \vdash (\text{throw } p \text{ to } k)$ prog
		$\Delta;\ \Gamma \vdash k\#p$ (in Harper)

$$\Delta ::= u_1 \div \phi_1, \cdots, u_n \div \phi_m$$
$$\Gamma ::= x_1 : \phi_1, \cdots, x_n : \phi_n$$

Rules

Structural

$$\frac{x : \phi \in \Gamma}{\Delta; \Gamma \vdash x : \phi} \qquad \frac{u \div \phi \in \Delta}{\Delta; \Gamma \vdash u \div \phi}$$

Truth

$$\frac{}{\Delta; \Gamma \vdash \langle\rangle : \top}$$

no refutation

False

no proof

$$\frac{}{\Delta; \Gamma \vdash \text{abort} - \div \bot}$$

Rules

Conjunction

$$\frac{\Delta; \Gamma \vdash p_1 : \phi_1 \qquad \Delta; \Gamma \vdash p_2 : \phi_2}{\Delta;\ \Gamma \vdash \langle p_1, p_2 \rangle : \phi_1 \wedge \phi_2}$$

$$\frac{\Delta; \Gamma \vdash k \div \phi_1}{\Delta; \Gamma \vdash \mathsf{fst}{-}; k \div \phi_1 \wedge \phi_2} \qquad \frac{\Delta; \Gamma \vdash k \div \phi_2}{\Delta; \Gamma \vdash \mathsf{snd}{-}; k \div \phi_1 \wedge \phi_2}$$

Disjunction

$$\frac{\Delta; \Gamma \vdash p_1 : \phi_1}{\Delta; \Gamma \vdash \mathsf{inl}\ p_1 : \phi_1 \vee \phi_2}$$

$$\frac{\Delta; \Gamma \vdash p_2 : \phi_2}{\Delta; \Gamma \vdash \mathsf{inr}\ p_2 : \phi_1 \vee \phi_2} \qquad \frac{\Delta; \Gamma \vdash k_1 \div \phi_1 \qquad \Delta; \Gamma \vdash k_2 \div \phi_2}{\Delta;\ \Gamma \vdash \mathsf{case}(-; k_1; k_2) \div \phi_1 \vee \phi_2}$$

Rules

Implication

$$\frac{\Delta; \Gamma,\ x : \phi_1 \vdash p_1 : \phi_2}{\Delta; \Gamma \vdash \lambda x : \phi_1 . p_1 : \phi_1 \supset \phi_2}$$

$$\frac{\Delta; \Gamma \vdash p_1 : \phi_1 \qquad \Delta; \Gamma \vdash k_2 \div \phi_2}{\Delta; \Gamma \vdash (-\ p_1); k \div \phi_1 \supset \phi_2}$$

Negation

$$\frac{\Delta; \Gamma \vdash p : \phi}{\Delta; \Gamma \vdash \mathbf{not}(p) \div \neg\phi} \qquad \frac{\Delta; \Gamma \vdash k \div \phi}{\Delta; \Gamma \vdash \mathbf{not}(k) : \neg\phi}$$

Rules

$$\frac{\Delta;\Gamma \vdash k \div \phi \qquad \Delta;\Gamma \vdash p : \phi}{\Delta;\Gamma \vdash (\text{throw } p \text{ to } k) \text{ prog}}$$

$$\frac{\Delta, u \div \phi;\Gamma \vdash (\text{throw } p \text{ to } k) \text{ prog}}{\Delta;\Gamma \vdash (\mathcal{C} u \div \phi.\text{throw } p \text{ to } k) : \phi}$$

$$\frac{\Delta;\Gamma, x : \phi \vdash (\text{throw } p \text{ to } k) \text{ prog}}{\Delta;\Gamma \vdash (\text{let } x :\ \phi = - \text{ in throw } p \text{ to } k) \div \phi}$$

Example

Proof of $(\phi \wedge (\psi \wedge \theta)) \supset (\theta \wedge \phi)$

$$
\begin{aligned}
&\lambda w : (\phi \wedge (\psi \wedge \theta)).\\
&\mathcal{C} u \div \theta \wedge \phi.\\
&\text{throw } w \text{ to } (\text{fst}-;\\
&\qquad \text{let } x : \ \phi = - \text{ in}\\
&\qquad \text{throw } w \text{ to } (\text{snd}-;\\
&\qquad\qquad \text{let } y : \ \psi \wedge \theta = - \text{ in}\\
&\qquad\qquad \text{throw } y \text{ to } (\text{snd}-;\\
&\qquad\qquad\qquad \text{let } z : \ \theta = - \text{ in}\\
&\qquad\qquad\qquad \text{throw } \langle z, x \rangle \text{ to } u)))
\end{aligned}
$$

Dynamics

$$\text{throw } \langle p_1, p_2 \rangle \text{ to } (\text{fst}-; k) \longrightarrow \text{throw } p_1 \text{ to } k$$

$$\text{throw } \langle p_1, p_2 \rangle \text{ to } (\text{snd}-; k) \longrightarrow \text{throw } p_2 \text{ to } k$$

$$\text{throw inl } p_1 \text{ to } (\text{case}(k_1; k_2)) \longrightarrow \text{throw } p_1 \text{ to } k_1$$

$$\text{throw inr } p_2 \text{ to } (\text{case}(k_1; k_2)) \longrightarrow \text{throw } p_2 \text{ to } k_2$$

$$\text{throw not}(k) \text{ to not}(p) \longrightarrow \text{throw } p \text{ to } k$$

$$\text{throw } \lambda x : \phi . p_2 \text{ to } (- \; p_1; k) \longrightarrow \text{throw } p_2[p_1/x] \text{ to } k$$

Dynamics

$$\text{throw } p_2 \text{ to } (\text{let } x:\ \phi = - \text{ in throw } p_1 \text{ to } k_1) \longrightarrow \text{throw } [p_2/x]p_1 \text{ to } [p_2/x]k_1$$

$$\text{throw } (\mathcal{C}\, u \div \phi.\text{throw } p_2 \text{ to } k_2) \text{ to } k_1 \longrightarrow \text{throw } [k_1/u]p_2 \text{ to } [k_1/u]k_2$$

$$\frac{}{(\text{throw } p \text{ to halt}) \text{ initial}} \qquad \frac{p \text{ canonical}}{(\text{throw } p \text{ to halt}) \text{ final}}$$

Another example

Peirce's Law: $((\phi \supset \psi) \supset \phi) \supset \phi$

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Peirce's Law: $((\phi \supset \psi) \supset \phi) \supset \phi$

$\lambda f : ((\phi \supset \psi) \supset \phi).$
$\mathcal{C} u \div \phi.$
throw $f\,(\lambda x : \phi.\text{throw } x \text{ to } u)$ to u

Another example

Excluded Middle: $\phi \vee \neg\phi$

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Excluded Middle: $\phi \vee \neg\phi$

$\mathcal{C}\, u \div \phi \vee \neg\phi.$
$\text{throw inr}(\text{not}(\text{let } x : \ \phi = - \text{ in throw inl } x \text{ to } u)) \text{ to } u$

Another example

Excluded Middle: $\phi \vee \neg\phi$

$\mathcal{C}\, u \div \phi \vee \neg\phi.$
$\text{throw inr}(\text{not}(\text{let } x : \ \phi = - \text{ in throw inl } x \text{ to } u)) \text{ to } u$

$$\text{throw em to case}(k_1; \text{not}(p_2)) \longrightarrow^{*} \text{throw } p_2 \text{ to } k_1$$

Double-Negation Translation

Every constructive proof is also a valid classical proof.

Double-Negation Translation

Every constructive proof is also a valid classical proof.

What about the inverse?

Double-Negation Translation

Classical	Constructive
$\Delta;\ \Gamma \vdash \phi$ true	$\neg\Delta^*;\ \Gamma^* \vdash \neg\neg\phi^*$ true
$\Delta;\ \Gamma \vdash \phi$ false	$\neg\Delta^*;\ \Gamma^* \vdash \neg\phi^*$ true
$\Delta;\ \Gamma \vdash \#$	$\neg\Delta^*;\ \Gamma^* \vdash \bot$ true

$$
\begin{array}{lcl}
\top^* & = & \top \\
\bot^* & = & \bot \\
(\phi_1 \wedge \phi_2)^* & = & \phi_1^* \wedge \phi_2^* \\
(\phi_1 \vee \phi_2)^* & = & \phi_1^* \vee \phi_2^* \\
(\phi_1 \supset \phi_2)^* & = & \phi_1^* \supset \neg\neg\phi_2^* \\
(\neg\phi)^* & = & \neg\phi^*
\end{array}
$$

Double-Negation Translation

- Computational meaning?

Double-Negation Translation

- Computational meaning?
- CPS translation

Double-Negation Translation

- Computational meaning?
- CPS translation
- Meaning of classical axioms?